The R Package fechner

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R and MATLAB

We present the R (http://www.r-project.org/) package **fechner** for Fechnerian scaling (FS) of object sets. Available on CRAN http://cran.r-project.org/package=fechner.

Other software for FS includes FSCAMDS, which runs on MATLAB, and a MATLAB toolbox. This software can be downloaded from, in respective order, http://www.psych.purdue.edu/∼ehtibar/ and http://www.psychologie.uni-oldenburg.de/stefan.rach/.

The finite, discrete version of FS, by far the most important for practical applications, is discussed in Dzhafarov and Colonius (2006). As any data set is necessarily finite, this is the version implemented in the package **fechner**.

Dzhafarov, E.N., & Colonius, H. (2006). Reconstructing distances among objects from their discriminability. *Psychometrika, 71*, 365–386.

Ünlü, A., Kiefer, T., & Dzhafarov, E.N. (2009). Fechnerian scaling in R: The package fechner. *Journal of Statistical Software, 31*(6), 1–24.

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ψ-Data

Let $\{x_1, \ldots, x_n\}$ be a set of objects endowed with a discrimination function $\psi(x_i, x_j)$. The primary meaning of $\psi(x_i, x_j)$ in FS is the probability with which x_i is judged to be different from x_j.

For example, a pair of colors (x_i, x_j) may be repeatedly presented to an observer (or a group of observers), and $\psi(x_i, x_j)$ may be estimated by the frequency of responses "they are different."

An empirical fact is that $\psi(x_i, x_j)$ is not a metric:

- $\psi(x_i, x_i)$ is not always zero;
- moreover, $\psi(x_i, x_i)$ and $\psi(x_j, x_j)$ for $i \neq j$ are not generally the same;
- $\psi(x_i, x_j)$ is generally different from $\psi(x_j, x_i)$;
- and the triangle inequality is not generally satisfied, $\psi(x_i, x_j) + \psi(x_j, x_k)$ may very well be less than $\psi(x_i, x_k)$.

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Regular Minimality

The only property of the ψ-data required by FS is regular minimality (RM):

- for every x_i there is one and only one x_j such that $\psi(x_i, x_j) < \psi(x_i, x_k)$ for all $k \neq j$ (this x_j is called the Point of Subjective Equality, PSE, of x_i);
- for every x_j there is one and only one x_i such that $\psi(x_i, x_j) < \psi(x_k, x_j)$ for all $k \neq i$ (this x_i is called the PSE of x_j);
- and x_j is the PSE of x_i if and only if x_i is the PSE of x_j.

Every data matrix in which the diagonal entry $\psi(x_i, x_i)$ is smaller than all entries $\psi(x_i, x_k)$ in its row $(k \neq i)$ and all entries $\psi(x_k, x_i)$ in its column $(k \neq i)$ satisfies RM in the simplest, so-called canonical, form. In this case every object x_i is the PSE of x_i. (Note that regular maximality can be defined analogously, replacing "minimal" with "maximal.")

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Canonical Relabeling

If RM is satisfied, the row objects (first observation area) and column objects (second observation area) can be presented in pairs of PSEs $(x_1, x_{k_1}), (x_2, x_{k_2}), \ldots, (x_n, x_{k_n})$, where $(k_1, k_2, \ldots, k_n)$ is a permutation of $(1, 2, \ldots, n)$.

FS identifies these PSE pairs and then relabels them so that two members of the same pair receive one and the same label:

$$(x_1, x_{k_1}) \mapsto (a_1, a_1), (x_2, x_{k_2}) \mapsto (a_2, a_2), \ldots, (x_n, x_{k_n}) \mapsto (a_n, a_n).$$

The relabeled and permuted matrix of ψ-data is a matrix in which each diagonal entry is minimal in its row and in its column. After this relabeling the original function $\psi(x_i, x_j)$ is redefined: $p_{ij} := \psi(a_i, a_j) := \psi\left(x_i, x_{k_j}\right)$.

In the package **fechner** the pairs of PSEs are assigned identical labels leaving intact the labeling of the rows and relabeling the columns with their corresponding PSEs. This is referred to as canonical relabeling.

Fechnerian Distance

For every pair of objects (a_i, a_j) consider all possible chains of objects $(a_i, a_{k_1}, \ldots, a_{k_r}, a_j)$, where $(a_{k_1}, \ldots, a_{k_r})$ is a sequence chosen from $\{a_1, \ldots, a_n\}$. For each such a chain compute what is called its psychometric length (of the first kind)

$$L^{(1)}(a_i, a_{k_1}, \ldots, a_{k_r}, a_j) = \sum_{m=0}^{m=r} \left(p_{k_m k_{m+1}} - p_{k_m k_m}\right),$$

where $a_i = a_{k_0}$ and $a_j = a_{k_{r+1}}$. The quantities $p_{k_m k_{m+1}} - p_{k_m k_m}$ are referred to as psychometric increments of the first kind.

Find a chain with the minimal value of $L^{(1)}$, and take this minimal value of $L^{(1)}$ for the quasidistance (quasimetric, or oriented metric) $G_{ij}^{(1)}$ from a_i to a_j (oriented Fechnerian distance of the first kind).

This quasimetric is symmetrized and transformed into a metric by computing $G_{ij}^{(1)} + G_{ji}^{(1)}$, and taking it for the overall Fechnerian distance G_{ij} between a_i and a_j.

Geodesic Chain, Geodesic Loop

Any chain $(a_i, a_{k_1}, \ldots, a_{k_r}, a_j)$ with $L^{(1)}(a_i, a_{k_1}, \ldots, a_{k_r}, a_j) = G_{ij}^{(1)}$ is called a geodesic chain (of the first kind).

The concatenation $(a_i, a_{k_1}, \ldots, a_{k_r}, a_j, a_{l_1}, \ldots, a_{l_s}, a_i)$ of a geodesic chain $(a_i, a_{k_1}, \ldots, a_{k_r}, a_j)$ and a geodesic chain $(a_j, a_{l_1}, \ldots, a_{l_s}, a_i)$ is called a geodesic loop.

The overall Fechnerian distance G_{ij} is the psychometric length (of the first kind) of a geodesic loop $(a_i, a_{k_1}, \ldots, a_{k_r}, a_j, a_{l_1}, \ldots, a_{l_s}, a_i)$, or equivalently, $(a_j, a_{l_1}, \ldots, a_{l_s}, a_i, a_{k_1}, \ldots, a_{k_r}, a_j)$.

Second Observation Area

One can also compute the psychometric length (of the second kind) of a chain $(a_i, a_{k_1}, \ldots, a_{k_r}, a_j)$ as

$$L^{(2)}(a_i, a_{k_1}, \ldots, a_{k_r}, a_j) = \sum_{m=0}^{m=r} \left(p_{k_{m+1}k_m} - p_{k_m k_m}\right),$$

where $p_{k_{m+1}k_m} - p_{k_m k_m}$ are called psychometric increments of the second kind. Define the quasidistance (oriented Fechnerian distance of the second kind) $G_{ij}^{(2)}$ from a_i to a_j as the minimal value of $L^{(2)}$ across all chains inserted between a_i and a_j.

It makes no difference for the final computation of the overall Fechnerian distance G_{ij}:

$$G_{ij} = G_{ij}^{(1)} + G_{ji}^{(1)} = G_{ij}^{(2)} + G_{ji}^{(2)}.$$

The $L^{(1)}$–length of any loop $(a_i, a_{k_1}, \ldots, a_{k_r}, a_j, a_{l_1}, \ldots, a_{l_s}, a_i)$ equals the $L^{(2)}$–length of the same loop traversed in the opposite direction, $(a_i, a_{l_s}, \ldots, a_{l_1}, a_j, a_{k_r}, \ldots, a_{k_1}, a_i)$.

S–Index, C–Index

The package **fechner** compares the value of G_{ij} to a generalized Shepardian index of dissimilarity (S–index) $S_{ij} = p_{ij} + p_{ji} - p_{ii} - p_{jj}$. Note that $G_{ij} \leq S_{ij}$ for all (a_i, a_j).

The comparison G_{ij} versus S_{ij} is of interest because it shows how different the psychometric increments $p_{ij} - p_{ii}$ are from an oriented metric. If $G_{ij} = S_{ij}$ for all (a_i, a_j), then the psychometric increments $p_{ij} - p_{ii}$ form an oriented metric, and the computation of G_{ij} is reduced to simple symmetrization: $(p_{ij} - p_{ii}) + (p_{ji} - p_{jj}) = S_{ij}$.

The greater the number of points (a_i, a_j) for which $G_{ij} < S_{ij}$ and the greater the differences $S_{ij} - G_{ij}$, the greater the "non-metricality" of the psychometric increments $p_{ij} - p_{ii}$. To quantify this "non-metricality" FS uses an ad hoc descriptive index (C–index)

$$C = \frac{2 \sum (S_{ij} - G_{ij})^2}{\sum S_{ij}^2 + \sum G_{ij}^2}.$$

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Functions, I: Main Function

The main function of the package is `fechner`:

```
fechner(X, format = c("probability.different", "percent.same", "general"),
        compute.all = FALSE, check.computation = FALSE)
```

The short computation returns a list, of the class `fechner`, containing such information as the pairs of PSEs, the canonical representation of the data in which regular minimality/maximality is satisfied in the canonical form and the rows and columns are canonically relabeled, the S–index, and most importantly, the overall Fechnerian distances and geodesic loops.

The long computation additionally yields intermediate results, such as the psychometric increments, the oriented Fechnerian distances, and the geodesic chains, and it also allows to check the equality $\left(G_{ij}^{(1)} + G_{ji}^{(1)}\right) - \left(G_{ij}^{(2)} + G_{ji}^{(2)}\right) = 0$.

Functions, II: Checking Properties

Regular minimality/maximality can be checked using the function `check.regular`:

```
check.regular(X, type = c("probability.different", "percent.same",
                          "reg.minimal", "reg.maximal"))
```

This function returns a list consisting of the canonical representation of the data, the pairs of PSEs, a character string saying which check was performed (regular minimality or regular maximality), and a logical indicating whether the original data are already in the canonical form.

The data format can be checked using the function `check.data`:

```
check.data(X, format = c("probability.different", "percent.same", "general"))
```

This function returns a matrix of the data with rows and columns labeled.

Functions, III: Plot, Print, and Summary Methods

`plot(x, level = 2)` graphs the results obtained in the FS analyses. It produces a scatterplot of the overall Fechnerian distance G versus the S–index, with rugs added to the axes and jittered to accommodate ties in the S–index and G values. The level of comparison refers to the minimum number of links in geodesic loops for the pairs of stimuli considered for the comparison.

`print(x)` prints the main results obtained in the FS analyses, which are the overall Fechnerian distances and the geodesic loops.

`summary(object, level = 2)` outlines the results obtained in the FS analyses. It returns a list consisting of the pairs of objects and their corresponding S–index and G values, the value of the Pearson correlation coefficient between them, the value of the C–index, and the level of comparison chosen.

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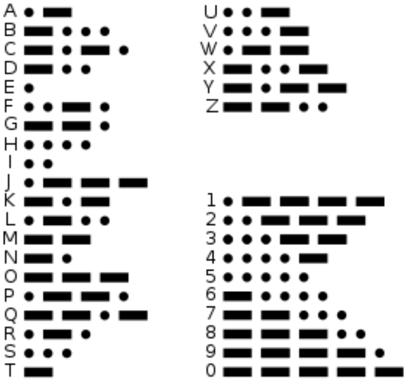
Morse Code Data, I

`morse`: Rothkopf's (1957) Morse code data of discrimination probabilities among 36 auditory Morse code signals for the letters $A, B, \ldots, Z$ and the digits $0, 1, \ldots, 9$. The `morse` data frame consists of 36 rows and 36 columns, representing the Morse code signals presented first and second, respectively. Each number in the data frame gives the percentage of subjects who responded "same" (choosing between "same" and "different") to the row signal followed by the column signal.



Morse code letters and digits

Rothkopf, E.Z. (1957). A measure of stimulus similarity and errors in some paired-associate learning tasks. *Journal of Exp. Psychology, 53*, 94–101.

Morse Code Data, II

For typographic reasons, we consider the 10-code subspace of the 36 Morse codes consisting of the codes for the letter B and the digits $0, 1, 2, 4, 5, \ldots, 9$.

```
R> indices <- which(is.element(names(morse), c("B", c(0, 1, 2, 4:9))))
R> (morse.subspace <- morse[indices, indices])
   B  1  2  4  5  6  7  8  9  0
B 84 12 17 40 32 74 43 17  4  4
1  5 84 63  8 10  8 19 32 57 55
2 14 62 89 20  5 14 20 21 16 11
4 19  5 26 89 42 44 32 10  3  3
5 45 14 10 69 90 42 24 10  6  5
6 80 15 14 24 17 88 69 14  5 14
7 33 22 29 15 12 61 85 70 20 13
8 23 42 29 16  9 30 60 89 61 26
9 14 57 39 12  4 11 42 56 91 78
0  3 50 26 11  5 22 17 52 81 94
```

The discrimination probabilities violate constant self-dissimilarity (e.g., digit 1 judged different from itself by 16%, but only by 6% for digit 0). Symmetry is violated as well (e.g., digits 4 and 5 judged to be different in 58% when 4 presented first, but in only 31% when 4 presented second).

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Checking Regular Minimality/Maximality

This part of the `morse` data satisfies regular maximality in the canonical form:

```
R> check.regular(morse.subspace, type = "percent.same")$check
[1] "regular maximality"
R> check.regular(morse.subspace, type = "percent.same")$in.canonical.form
[1] TRUE
```

The data set `noRegMin` (artificial data set included in the package) satisfies neither regular minimality nor regular maximality:

```
R> check.regular(noRegMin, type = "reg.minimal")
regular minimality is violated: entry in row #1 and column #10
is minimal in row #1 but not in column #10

R> check.regular(noRegMin, type = "reg.maximal")
regular maximality is violated: entry in row #2 and column #6
is maximal in row #2 but not in column #6
```

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FS Analysis using Short Computation, I

The function `fechner` is the main function of the package and provides the FS computations. For instance, the overall Fechnerian distances using short computation (`compute.all = FALSE`) are:

```
R> f.scal.subspace.mo <- fechner(morse.subspace,
R+ format="percent.same",compute.all=FALSE,check.computation=FALSE)
R> f.scal.subspace.mo$overall.Fechnerian.distances
     B    1    2    4    5    6    7    8    9    0
B 0.00 1.51 1.42 0.97 0.97 0.18 0.61 1.05 1.49 1.60
1 1.51 0.00 0.48 1.60 1.50 1.49 1.27 0.99 0.61 0.73
2 1.42 0.48 0.00 1.32 1.64 1.49 1.25 1.28 1.06 1.21
4 0.97 1.60 1.32 0.00 0.68 0.97 1.27 1.45 1.65 1.69
5 0.97 1.50 1.64 0.68 0.00 1.08 1.39 1.60 1.71 1.74
6 0.18 1.49 1.49 0.97 1.08 0.00 0.43 0.87 1.35 1.46
7 0.61 1.27 1.25 1.27 1.39 0.43 0.00 0.44 0.92 1.18
8 1.05 0.99 1.28 1.45 1.60 0.87 0.44 0.00 0.63 0.83
9 1.49 0.61 1.06 1.65 1.71 1.35 0.92 0.63 0.00 0.26
0 1.60 0.73 1.21 1.69 1.74 1.46 1.18 0.83 0.26 0.00
```

FS Analysis using Short Computation, II

The information provided using the short computation, an overview:

```
R> attributes(f.scal.subspace.mo)
$names
[1] "points.of.subjective.equality"   "canonical.representation"
[3] "overall.Fechnerian.distances"    "geodesic.loops"
[5] "graph.lengths.of.geodesic.loops" "S.index"

$computation
[1] "short"

$class
[1] "fechner"
```

FS Analysis using Long Computation

An overview of the information computed under the long computation (`compute.all = TRUE`), which additionally yields intermediate results and also allows for a check of computations:

```
R> f.scal.subspace.long.mo <- fechner(morse.subspace,
R+ format="percent.same",compute.all=TRUE,check.computation=TRUE)
R> attributes(f.scal.subspace.long.mo)
$names
 [1] "points.of.subjective.equality"      "canonical.representation"
 [3] "psychometric.increments.1"          "psychometric.increments.2"
 [5] "oriented.Fechnerian.distances.1"    "overall.Fechnerian.distances.1"
 [7] "oriented.Fechnerian.distances.2"    "overall.Fechnerian.distances.2"
 [9] "check"                              "geodesic.chains.1"
[11] "geodesic.loops.1"                   "graph.lengths.of.geodesic.chains.1"
[13] "graph.lengths.of.geodesic.loops.1"  "geodesic.chains.2"
[15] "geodesic.loops.2"                   "graph.lengths.of.geodesic.chains.2"
[17] "graph.lengths.of.geodesic.loops.2"  "S.index"

$computation
[1] "long"

$class
[1] "fechner"
```

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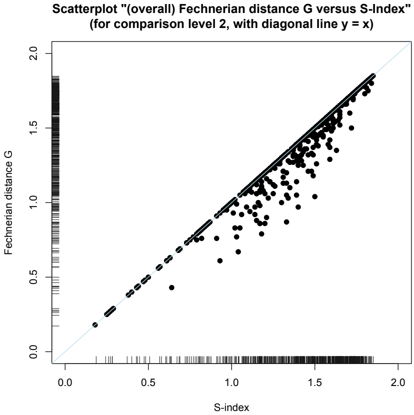
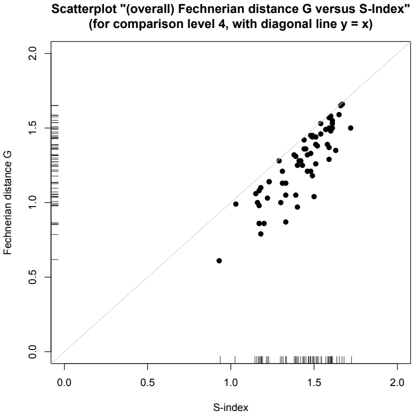
Plotting

Plotting the `fechner` object `f.scal.morse` (computed based on the entire Morse code data set) gives scatterplots (for comparison levels 2 and 4, respectively):

```
R> plot(f.scal.morse)
```




```
R> plot(f.scal.morse, level = 4)
```



Summarizing

The corresponding summary of the `fechner` object `f.scal.morse`, including the Pearson correlation coefficient and the C–index:

```
R> summary(f.scal.morse)

number of stimuli pairs used for comparison: 630

summary of corresponding S-index values:
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  0.180   1.260   1.520   1.435   1.670   1.850

summary of corresponding Fechnerian distance G values:
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  0.180   1.203   1.490   1.405   1.660   1.850

Pearson correlation: 0.9764753

C-index: 0.002925355

comparison level: 2
```

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Fechnerian Scaling in R

By contributing the package **fechner** in R we hope to have established a basis for computational work in this field. Interactive visualization and computational statistics approaches can be utilized in post-Fechnerian analyses to make the results obtained by Fechnerian scaling more explorable and interpretable.

The realization of Fechnerian scaling in R may also prove valuable in applying current or conventional statistical methods to the theory of Fechnerian scaling. For instance, the determination of confidence regions (e.g., for overall Fechnerian distances) and hypothesis testing (e.g., testing for RM) in Fechnerian scaling are likely to be based on resampling methods. Such an endeavor would involve extensive computer simulation, something R would be ideally suited for.

The package **fechner** will have to be extended to incorporate such approaches.