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Mixtures of Rasch Models

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`http://www.uibk.ac.at/statistics/`

Introduction

- Rasch model for measuring latent traits
- Model assumption: Item parameters estimates do not depend on person sample
- Violated in case of differential item functioning (DIF)
- Several approaches to test for DIF:
  - LR tests, Wald tests
  - Rasch trees
  - Mixture models
- Here: Two versions of the mixture model approach

Rasch Model

Probability for person i to solve item j:

$$P(Y_{ij} = y_{ij} | \theta_i, \beta_j) = \frac{e^{y_{ij}(\theta_i - \beta_j)}}{1 + e^{\theta_i - \beta_j}}$$

- y_{ij}: Response by person i to item j
- θ_i: Ability of person i
- β_j: Difficulty of item j

ML Estimation

Factorization of the full likelihood on basis of the scores $r_i = \sum_{j=1}^{m} y_{ij}$

$$\begin{aligned} L(\boldsymbol{\theta}, \boldsymbol{\beta}) &= f(\mathbf{y} | \boldsymbol{\theta}, \boldsymbol{\beta}) \\ &= h(\mathbf{y} | \boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\beta}) g(\boldsymbol{r} | \boldsymbol{\theta}, \boldsymbol{\beta}) \\ &= h(\mathbf{y} | \boldsymbol{r}, \boldsymbol{\beta}) g(\boldsymbol{r} | \boldsymbol{\theta}, \boldsymbol{\beta}) \end{aligned}$$

- Joint ML: Joint estimation of $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ is inconsistent
- Marginal ML: Assume distribution for $\boldsymbol{\theta}$ and integrate out in $g(\boldsymbol{r} | \boldsymbol{\theta}, \boldsymbol{\beta})$
- Conditional ML: Assume $g(\boldsymbol{r}) = g(\boldsymbol{r} | \boldsymbol{\theta}, \boldsymbol{\beta})$ as given or that it does not depend on $\boldsymbol{\theta}, \boldsymbol{\beta}$ (but potentially other parameters). Hence, $g(\boldsymbol{r})$ is a nuisance term and only $h(\boldsymbol{y} | \boldsymbol{r}, \boldsymbol{\beta})$ needs to be maximized.

Mixture Models

- Mixture models are a tool to model data with unobserved heterogeneity caused by, e.g., (latent) groups
- Mixture density = $\sum$ weight $\times$ component
- Weights are a priori probabilities for the components
- Components are densities or (regression) models

Mixtures of Rasch Models

- Mixture of the full likelihoods by Rost (1990):

$$f(\mathbf{y}|\boldsymbol{\pi}, \boldsymbol{\psi}, \boldsymbol{\beta}) = \prod_{i=1}^{n} \sum_{k=1}^{K} \pi_k \psi_{r_i,k} h(\mathbf{y}_i | r_i, \boldsymbol{\beta}_k)$$

  with $\psi_{r_i,k} = g_k(r_i)$

- Mixture of the conditional likelihoods:

$$f(\mathbf{y}|\boldsymbol{\pi}, \boldsymbol{\beta}) = \prod_{i=1}^{n} \sum_{k=1}^{K} \pi_k h(\mathbf{y}_i | r_i, \boldsymbol{\beta}_k)$$

Parameter Estimation

EM algorithm by Dempster, Laird and Rubin (1977)

- Group membership is seen as a missing value
- Optimization is done iteratively by alternate estimation of group membership (E-step) and component densities (M-step)
- E-step:

$$\hat{p}_{ik} = \frac{\hat{\pi}_k h(\mathbf{y}_i | r_i, \hat{\boldsymbol{\beta}}_k)}{\sum_{g=1}^{K} \hat{\pi}_g h(\mathbf{y}_i | r_i, \hat{\boldsymbol{\beta}}_g)}$$

- M-step:
  For each component separately

$$\hat{\boldsymbol{\beta}}_k = \underset{\beta_k}{\operatorname{argmax}} \sum_{i=1}^{n} \hat{p}_{ik} \log h(\mathbf{y}_i | r_i, \hat{\boldsymbol{\beta}}_k)$$

Number of Components

How can the number of components k be established?

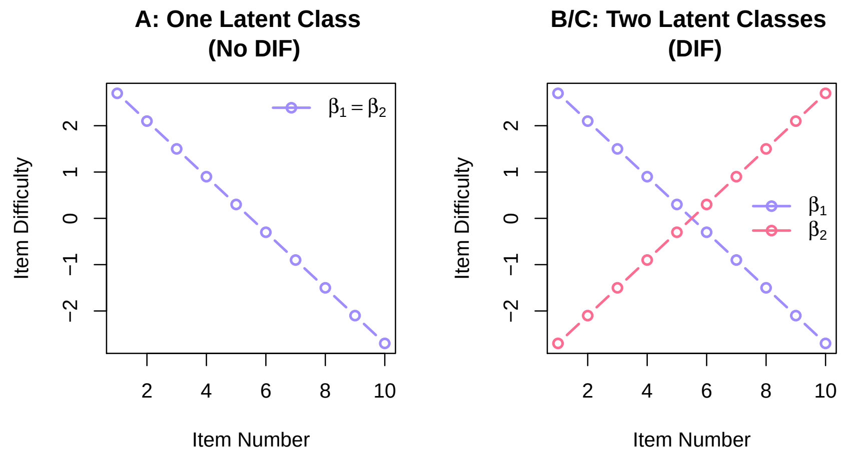
- A priori known number of groups in the data
- LR test: Regularity conditions are not fulfilled
  $\rightarrow$ Distribution under H_0 unknown
  $\rightarrow$ Bootstrap necessary
- Information criteria: AIC, BIC, ICL

Simulation Design

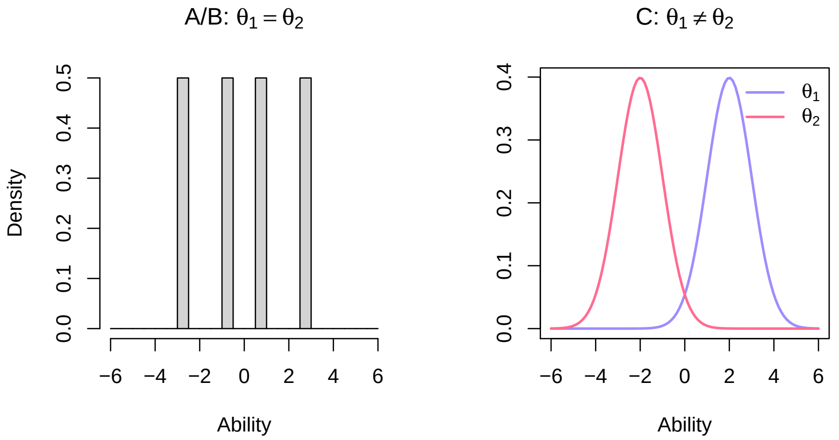
- 10 items, 1800 people, equal group sizes
- Latent groups in item and/or person parameters:

	$\beta_1 = \beta_2$	$\beta_1 \neq \beta_2$
$\theta_1 = \theta_2$	A	B
$\theta_1 \neq \theta_2$		C

Item Parameters



Person Parameters



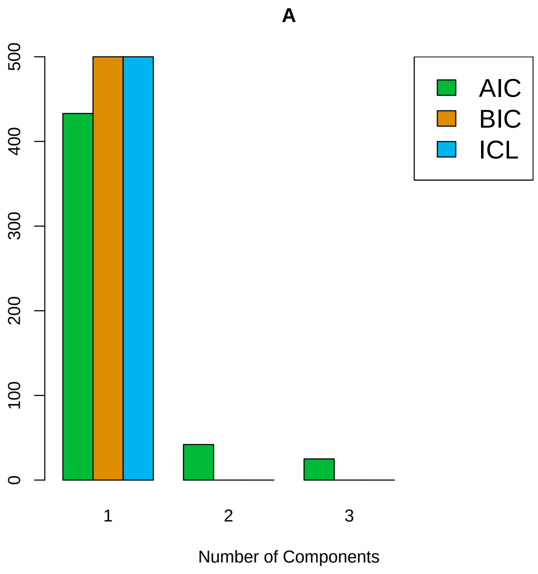
Criteria for Goodness of Fit

- Number of components
- Rand index:

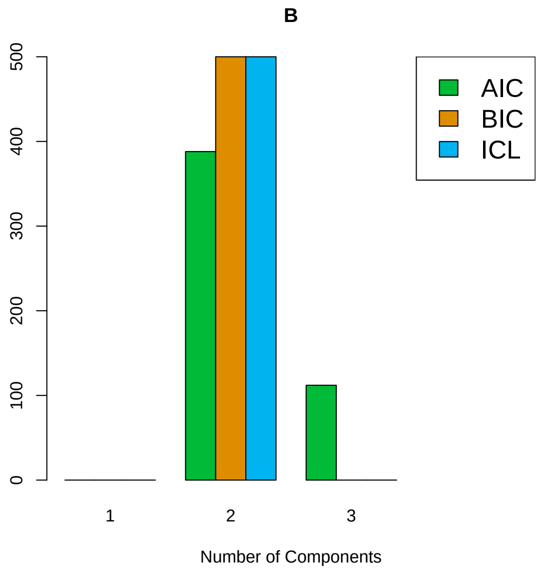
  Agreement between true and estimated partition
- Mean residual sum of squares:

  Agreement between true and estimated (item) parameter vector

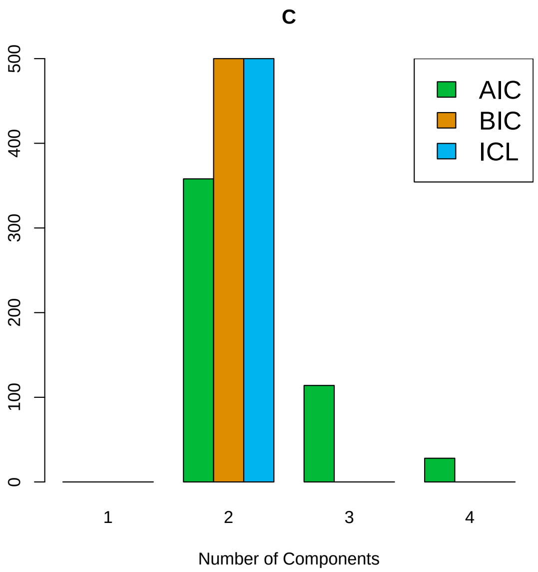
No Latent Classes (No DIF)



Two Latent Classes (DIF)



Latent Structure in Item and Person Parameters (DIF + Ability Differences)



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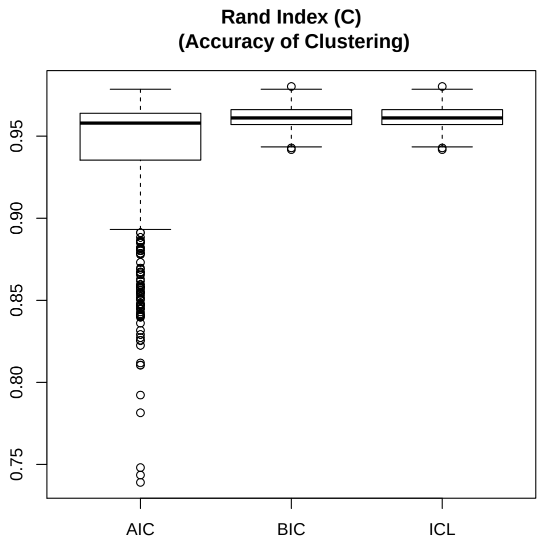

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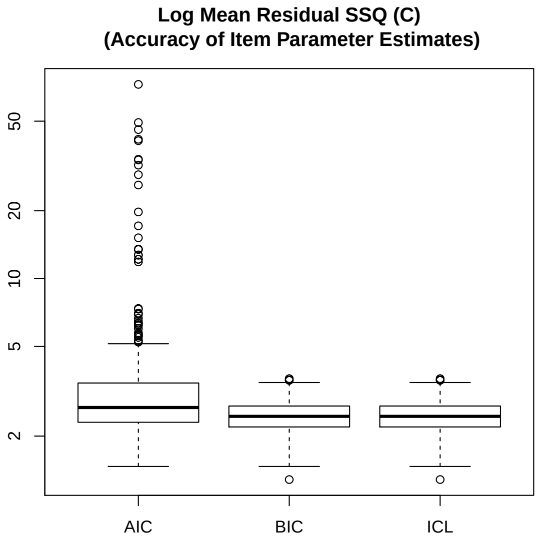

Summary and Outlook

- Model suitable for detecting latent classes with DIF
- Model also suitable when a latent structure in the person parameters is present
- AIC tends to overestimate the correct number of classes, BIC and ICL work well
- Clustering of the observations works well
- Estimation of the item parameters in the components works reasonably well
- Comparison with Rost's MRM to follow

Literature

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